June 26^{th}, 2024.

Review.

§7.8. Improper Integrals.

1) $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$, $\int_{-\infty}^\infty f(x)\,dx$.

2) $\int_a^b f(x)\,dx$, but f has discontinuities between a and b.

1) $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \underbrace{\int_a^t f(x)\,dx}_{\text{we know this definite integral}}.$$

$\int_a^\infty f(x)\,dx$ is convergent if the limit above exists.
otherwise, it is divergent.

Converge: $\int_1^\infty \frac{1}{x^2}\,dx$, $\int_1^\infty \frac{1}{x^p}$ $(p>1)$

$\int_0^\infty \frac{1}{e^x}\,dx$

2) $\int_a^b f(x)\,dx$ with f has discontinuity at $c \in [a,b]$.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

write. $\int_a^c f(x)\,dx = \lim_{t\to c} \int_a^t f(x)\,dx$.

$\int f(x)\,dx$ ... $t\to c$ ... t

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$$\int_a f(x)\,dx \quad \lim_{t\to c} \int_a^t f(x)\,dx.$$

$$\int_c^b f(x)\,dx = \lim_{t\to c} \int_t^b f(x)\,dx.$$

Example: Determine whether the integral is convergent or divergent.

1) $\int_1^\infty 2x^{-3}\,dx.$ (I)

$$\int_1^\infty 2x^{-3}\,dx = \lim_{t\to\infty} \int_1^t 2x^{-3}\,dx.$$

$$= \lim_{t\to\infty} \frac{2x^{-2}}{-2}\Big|_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{x^2}\Big|_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t^2} - \frac{1}{-1}\right)$$

$$= 1$$

$\Rightarrow$ The improper integral is convergent, has value 1.

$$\int_1^\infty \frac{1}{x^2+4}\,dx.$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4}\,dx$$

Comparison test. $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent.

and $x^2+4 > x^2$

$$\Rightarrow \frac{1}{x^2+4} < \frac{1}{x^2}$$

$$\Rightarrow \int_1^\infty \frac{1}{x^2+4}\,dx < \int_1^\infty \frac{1}{x^2}\,dx$$

convergen

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converges

$$\Rightarrow \int_1^{\infty} \frac{1}{x^2+4}\,dx \text{ converges}$$

Formula: $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

$$\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+2^2}$$

$$= \lim_{t\to\infty} \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_1^t$$

$$= \frac{1}{2}\lim_{t\to\infty}\left(\arctan\left(\frac{t}{2}\right) - \arctan\left(\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \arctan\frac{1}{2}\right)$$

c) $\displaystyle\int_{-2}^{\infty} \frac{1}{x+4}\,dx.$

$$= \lim_{t\to\infty} \int_{-2}^t \frac{1}{x+4}\,dx.$$

$$= \lim_{t\to\infty} \ln|x+4|\,\Big|_{-2}^t$$

$$= \lim_{t\to\infty} \left(\ln|t+4| - \ln|-2+4|\right)$$

$$= \infty$$

The improper integral is divergent.

d) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx.$

$\boxed{\dfrac{\ln x}{x^2} < \dfrac{x^{1/2}}{x^2}}$ when x large $\leftarrow$ we have.

$$\ln x - x^{1/2} < 0 \iff \ln x < x^{1/2}.$$

x^β ; $\beta < 0$.

Also $g(x) = \frac{x^{1/2}}{x^2} = \frac{1}{x^{3/2}}$ $\left(\frac{3}{2} > 1\right)$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$.

$\Rightarrow \int_1^\infty \frac{1}{x^{3/2}}\,dx$ is convergent.

$\Rightarrow \int_1^\infty \frac{\ln x}{x^2}\,dx < \int_1^\infty \frac{1}{x^{3/2}}\,dx$ | by (comparison test)

and converges.

e) $\int_1^\infty \frac{\ln x}{x}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx$

Integration by parts

$\begin{cases} u = \ln x \\ dv = \frac{1}{x}\,dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}\,dx \\ v = \ln x \end{cases}$

$\int_1^t \frac{\ln x}{x}\,dx = \ln^2 x\Big|_1^t - \int_1^t \ln x \frac{1}{x}\,dx.$

Substitution $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$\int_1^t \frac{\ln x}{x}\,dx = \int_0^{\ln t} u\,du = \frac{u^2}{2}\Big|_0^{\ln t} = \frac{\ln^2 t}{2}$

$\begin{cases} x = 1 \\ x = t \end{cases} \Rightarrow \begin{cases} u = 0 \\ u = \ln t \end{cases}$

$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \frac{\ln^2 t}{2} = \infty$

f) $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \lim \int^t \frac{x}{\sqrt{\ldots}}\,dx$

does not exist at 1

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Substitution. $u = 1 - x^2 \Rightarrow du = -2x\,dx$

$\Rightarrow dx = \dfrac{du}{-2x}$

$$\begin{cases} x = 0 \\ x = t \end{cases} \Rightarrow \begin{cases} u = 1 \\ u = 1 - t^2 \end{cases}$$

$$\int_0^t \frac{x\,dx}{\sqrt{1-x^2}} = \int_1^{1-t^2} \frac{x\,du}{\sqrt{u}\,(-2x)} = -\frac{1}{2}\int_1^{1-t^2} \frac{du}{\sqrt{u}}$$

$$= -u^{1/2}\Big|_1^{1-t^2}$$

$$= -\sqrt{1-t^2} + 1.$$

$$\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\left(-\sqrt{1-t^2} + 1\right) = 1.$$

0

g) $\displaystyle\int_{-5}^{5} \frac{1}{x^{2/3}}\,dx$

$\rightarrow$ does not define at $x = 0$. $\Rightarrow$ Type 2.

$$= \int_{-5}^{0} \frac{1}{x^{2/3}}\,dx + \int_0^5 \frac{1}{x^{2/3}}\,dx$$

$$= I + II$$

Compute I: $\displaystyle\int_{-5}^{0} \frac{1}{x^{2/3}}\,dx = \lim_{t\to 0^-}\int_{-5}^{t} \frac{1}{x^{2/3}}\,dx$ $\qquad x^{-\frac{2}{3}}$

$$= \lim_{t\to 0^-} \frac{x^{\frac{1}{3}}}{\frac{1}{3}}\Big|_{-5}^{t}$$

$$= \lim_{t\to 0^-} 3\left(t^{\frac{1}{3}} - (-5)^{\frac{1}{3}}\right)$$

$$= 3.\left(5^{\frac{1}{3}}\right)$$

$\Rightarrow$ Improper integral I converges

that

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Compute I: $\int_{-5}^{0}\frac{1}{x^{2/3}}\,dx = \lim_{t\to 0^-}\int_{-5}^{t}\frac{1}{x^{2/3}}\,dx$ $\quad x^{-\frac{2}{3}}$

$= \lim_{t\to 0^-} \left.\frac{x^{\frac{1}{3}}}{\frac{1}{3}}\right|_{-5}^{t}$

$= \lim_{t\to 0^-} 3\left(t^{\frac{1}{3}} - (-5)^{\frac{1}{3}}\right)$

$= 3.\left(5^{\frac{1}{3}}\right)$

$\Rightarrow$ Improper integral I converges

Note that $\int_{0}^{5}\frac{1}{x^{2/3}}\,dx = \int_{-5}^{0}\frac{1}{x^{2/3}}\,dx = 3.\,5^{\frac{1}{3}}$

$\Rightarrow \int_{-5}^{5}\frac{1}{x^{2/3}}\,dx$ is convergent.

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$$\int_1^\infty \frac{1}{e^x x^2}\,dx$$

$\frac{1}{e^x x^2} < \frac{1}{x^2}$ so by Comparison test. $\int_1^\infty \frac{1}{x^2}\,dx$ converges

$\rightarrow \int_1^\infty \frac{1}{e^x x^2}\,dx$ is convergent.

2) $\int_1^\infty \frac{x}{\sqrt{x^6+5}}\,dx$ $\qquad \frac{x}{\sqrt{x^6+5}} < \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$

converges

3) $\int_1^\infty \frac{10+\sin x}{\sqrt{x+2}}\,dx$ $\qquad \frac{10+\sin x}{\sqrt{x+2}} > \frac{5}{\sqrt{x+2}}$

$\int_1^\infty \frac{5}{\sqrt{x+2}}\,dx$ diverges.

$\rightarrow$ diverges.

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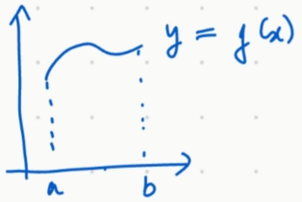
§ 8.1 Arc length.



$$\text{Length} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$
$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Length of curve $x = g(y)$

$$\text{Length} = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

Example: $36y^2 = (x^2-4)^3 \quad 2 \le x \le 3, \ y \ge 0$

$$y = \sqrt{\frac{(x^2-4)^3}{36}} = \frac{1}{6}(x^2-4)^{\frac{3}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{6}\,\frac{3}{\cancel{2}}\,(x^2-4)^{\frac{1}{2}}\,\cancel{2}x = \frac{1}{2}\,(x^2-4)^{\frac{1}{2}}x.$$

$$\text{Length} = \int_2^3 \sqrt{1+\frac{1}{4}(x^2-4)x^2}\,dx.$$
$$= \int_2^3 \sqrt{1+\frac{1}{4}x^4 - x^2}\,dx.$$

Complete the square. $(A+B)^2 = A^2+2AB+B^2$

$$1+\left(\frac{1}{4}x^4 - x^2\right) = 1+\left(\left(\frac{1}{2}x^2\right)^2 - 2\cdot\frac{1}{2}x^2 + 1 - 1\right)$$
$$A^2 - 2A + B^2 - B^2$$
$$= 1 + \left(\left(\frac{1}{2}x^2\right)^2 - 2\cdot\frac{1}{2}x^2 + 1\right) - 1.$$
$$= \left(\frac{1}{2}x^2 - 1\right)^2$$

$$= \int_2^3 \sqrt{\left(\frac{1}{2}x^2 - 1\right)^2}\,dx$$

$$= \int \left(\frac{1}{2}x^2 - 1\right)dx$$

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Complete the square.

$$(A+B)^2 = A^2+2AB+B^2$$

$$1+\left(\frac{1}{4}x^4-x^2\right) = 1+\left(\left(\frac{1}{2}x^2\right)^2-2\cdot\frac{1}{2}x^2+1-1\right)$$

$$A^2 - 2A + B^2 - B^2$$

$$= 1+\left(\left(\frac{1}{2}x^2\right)^2-2\cdot\frac{1}{2}x^2+1\right)-1.$$

$$= \left(\frac{1}{2}x^2-1\right)^2$$

$$= \int_2^3 \sqrt{\left(\frac{1}{2}x^2-1\right)^2}\,dx$$

$$= \int_2^3 \left(\frac{1}{2}x^2-1\right)dx$$

$$= \left(\frac{1}{2}\frac{x^3}{3}-x\right)\Big|_2^3.$$

Part A: Midterms 1 + 2 (Review) 100 p.

1) Curve sketching 20 p.
2) Optimization 20p
3) 5 integrals: (substitution, trigonometric, integration by parts) 50p
4) area between curve. 10 p.

Part B.

1) 40p: integrals: trigonometric substitution,
2) 30p: integration of rational functions
3) 10p arc length.
4) 20p: Improper integrals.

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